Continuous Functions

The concept of a continuous function is very important in analysis. Almost every elementary functions are continuous. In fact, continuity of a function is crucial for us to "draw" its graph.

Definition (c.f. Definition 5.1.1 & 5.1.5)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. f is said to be *continuous* at $c \in A$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \quad \text{whenever } |x - c| < \delta \text{ and } x \in A.$$

Moreover, f is said to be *continuous on* A if f is continuous at every $c \in A$.

Remark. Compare it to the definition of limit of functions. They are very similar, be careful about the condition on the point c.

- For limit, c is **required** to be a cluster point of A but c **need not** lies in A.
- For continuity, c is **required** to lie in A but c **need not** be a cluster point of A.

Hence if c is a cluster point in A. Then f is continuous at c if and only if

$$f(c) = \lim_{x \to c} f(x).$$

Sequential Criterion for Continuity (c.f. 5.1.3)**.** *A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Disontinuity Criterion (c.f. 5.1.4)**.** *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.*

Exercise. Prove one of these theorems (they are equivalent).

Remark. Since c may not be a cluster point of A, you cannot directly apply the **Sequential Criterion for Limits of Functions** or the **Divergence Criteria for Limits of Functions**, but the proofs are similar.

Example 1. Consider $A = \mathbb{N}$, then any functions $f : \mathbb{N} \to \mathbb{R}$ is continuous.

Proof. Let $c \in \mathbb{N}$ and $\varepsilon > 0$. Take $\delta = 1$. Then whenever $|x - c| < \delta = 1$ and $x \in \mathbb{N}$, it implies that $x = c$. Hence

$$|f(x) - f(c)| = 0 < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that f is continuous at c. Since $c \in \mathbb{N}$ is arbitrary, it follows that f is continuous on $\mathbb{N}$. □

Exercise. Let $A \subseteq \mathbb{R}$ be a finite set. Show that any functions $f : A \to \mathbb{R}$ is continuous.

Example 2 (c.f. Example 5.1.6(a)-(e))**.** Most of the elementary functions we learn are continuous on their **maximum domains of definition**, the following are a few examples:

(a) The constant function $f_1(x) = b$.

(b) The identity map $f_2(x) = x$.

(c) The square function $f_3(x) = x^2$.

(d) The reciprocal function $f_4(x) = 1/x$.

Proof. Let's show that f_4 is continuous on $\mathbb{R} \setminus \{0\}$. Note that for any $x, c \neq 0$,

$$|f_4(x) - f_4(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{1}{|x||c|}|x - c|.$$

If $c > 0$, note that if $|x - c| < \frac{1}{2}c$, then $0 < \frac{1}{2}c < x < \frac{3}{2}c$. In this case,

$$\frac{1}{|x||c|}|x - c| = \frac{1}{cx}|x - c| < \frac{2}{c^2}|x - c|.$$

Let $\varepsilon > 0$. Take $\delta = \min\{\frac{1}{2}c, \frac{c^2}{2}\varepsilon\}$. Then whenever $|x - c| < \delta$,

$$|f_4(x) - f_4(c)| = \frac{1}{|x||c|}|x - c| < \frac{2}{c^2}|x - c| < \frac{2}{c^2}\delta \leq \varepsilon.$$

□

Exercise. Do the case for $c < 0$.

Example 3 (c.f. Example 5.2.3(c))**.** The sine function is continuous on $\mathbb{R}$.

Proof. We will use the fact that $|\sin z| \leq |z|$ for all $z \in \mathbb{R}$. Notice that for any $x, c \in \mathbb{R}$,

$$|\sin x - \sin c| = 2\left|\cos\left(\frac{x + c}{2}\right)\right|\left|\sin\left(\frac{x - c}{2}\right)\right| \leq |x - c|.$$

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Take $\delta = \varepsilon > 0$. then whenever $|x - c| < \delta$,

$$|\sin x - \sin c| \leq |x - c| < \delta = \varepsilon.$$

The result follows. □

Exercise. Show that the cosine function is continuous on $\mathbb{R}$.

Example 4 (c.f. Example 5.1.6(g))**.** Consider the Dirichlet's function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then h is not continuous at every point $c \in \mathbb{R}$.

Proof. Suppose $c \in \mathbb{Q}$. We apply the **Discontinuity Criterion**. By the density of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, there is a sequence (x_n) of irrational numbers that converges to c. Hence $h(x_n) = 0$ for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} h(x_n) = 1 \neq 0 = h(c).$$

It follows that h is discontinuous at c. □

Exercise. Do the case for $c \in \mathbb{R} \setminus \mathbb{Q}$.

Example 5 (c.f. Section 5.1, Ex.7)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there is a $\delta > 0$ such that $f(x) > 0$ whenever $x \in (c - \delta, c + \delta)$.

Remark. Geometrically, it means that if a continuous function f takes a positive value at c, then f is positive on a neighbourhood of c.

Solution. Take $\varepsilon = f(c)/2 > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}, \quad \text{whenever } |x - c| < \delta.$$

Hence if $x \in (c - \delta, c + \delta)$, i.e., $|x - c| < \delta$,

$$0 < \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c).$$

In particular, $f(x) > 0$.

Example 6 (c.f. Section 5.2, Ex.8)**.** Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

Solution. The answer to this question is positive. By considering the continuous function $f - g$, we can assume g is the zero function. It remains to show that if $f(r) = 0$ for all $r \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.
Suppose on a contrary that $f(x) \neq 0$ for some $x \in \mathbb{R}$. If $f(x) > 0$, the previous example yields $\delta > 0$ such that

$$f(y) > 0, \quad \text{whenever } y \in (x - \delta, x + \delta).$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find some rational number $r \in (x - \delta, x + \delta)$. It follows that $0 = f(r) > 0$, which is a contradiction. Similarly, we can find a contradiction if $f(x) < 0$. It follows that $f(x) = 0$ for all $x \in \mathbb{R}$.

Exercise. Prove the same result by using the **Sequential Criterion for Continuity**.